LOCALIZATIONS, COLOCALIZATIONS AND NON ADDITIVE $*$-OBJECTS

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Abstract. Given two arbitrary categories, a pair of adjoint functors between them induces three pairs of full subcategories, as follows: the subcategories of reflexive objects, that is objects for which the unit (respectively counit) of the adjunction is an isomorphism; the subcategories of local (respectively colocal) objects w.r.t. these adjoint functors; the subcategories of cogenerated (respectively generated) objects w.r.t this adjoint pair, namely objects for which the unit (counit) of the adjunction is a monomorphism (an epimorphism). We investigate some cases in which the subcategory of reflexive objects coincide with the subcategory of (co)local objects or with the subcategory of (co)generated objects. As an application we define and characterize (weak) $*$-objects in the non additive case, more precisely weak $*$-acts.

Introduction

In mathematics the concept of localization has a long history. The origin of the concept is the study of some properties of maps around a point of a topological space. In the algebraic sense, localization provides a method to invert some morphisms in a category. Making abstraction of some technical set theoretic problems, given a class of morphisms Σ in a category $\mathcal{A}$, there is a category $\mathcal{A}[\Sigma^{-1}]$ and a functor $\mathcal{A} \to \mathcal{A}[\Sigma^{-1}]$ universal with the property that it sends any morphism in Σ to an isomorphism. This functor will be called a *localization*, if it has a right adjoint, which will frequently be fully faithful. Dually this functor is called a *colocalization* provided that it has a left adjoint.

Let us present the organization and the main results of the paper. In the first section we set the notations, we define the main notions used throughout the paper and we record some easy properties concerning these notions.

In Section 2 are stated and proved the formal results. In Theorem 2.1 we investigate when a pair of adjoint functors induces equivalences between the subcategories of local and colocal objects. Next we obtain a formal characterization of a non additive (weak) $*$-object, that is Proposition 2.7 and Theorem 2.8. Concerning both characterizations note that only one implication was considered before, namely in [2].

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In Section 3 we define and characterize the notions of a (weak) $*$-act over a monoid, in Proposition 3.2 and Theorem 3.3, providing in this way a translation of the notion of (weak) $*$-module in this new setting. It is interesting to note that our approach may be continued by developing a theory analogous with so called tilting theory for modules. The Morita theory for monoids is a consequence of our results.

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1. Notations and preliminaries

All subcategories which we consider are full and closed under isomorphisms, so if we speak about a class of objects in a category we understand also the respective subcategory. For a category $\mathcal{A}$ we denote by $\mathcal{A}^{\to}$ the category of all morphisms in $\mathcal{A}$. We denote by $\mathcal{A}(-,-)$ the bifunctor assigning to any two objects of $\mathcal{A}$ the set of all morphisms between them.

Consider a functor $H : \mathcal{A} \to \mathcal{B}$. The *(essential) image* of H is the subcategory $\operatorname{Im} H$ of $\mathcal{B}$ consisting of all objects $Y \in \mathcal{B}$ satisfying $Y \cong H(X)$ for some $X \in \mathcal{A}$. In contrast we shall denote by $\operatorname{im} \alpha$ the categorical notion of *image* of a morphism $\alpha : X \to X'$ in $\mathcal{A}^{\to}$, that is the smallest subobject $i : X'' \to X'$ such that f factors through i (see [13, 1.18]). A morphism $\alpha \in \mathcal{A}^{\to}$ is called an *H-equivalence*, provided that $H(\alpha)$ is an isomorphism. We denote by $\operatorname{Eq}(H)$ the subcategory of $\mathcal{A}^{\to}$ consisting of all H-equivalences. An object $X \in \mathcal{A}$ is called *H-local* (*H-colocal*) if, for any H-equivalence ϵ, the induced map $\epsilon_* = \mathcal{A}(\epsilon, X)$ (respectively, $\epsilon^* = \mathcal{A}(X, \epsilon)$) is bijective, that means it is an isomorphism in the category $\mathcal{S}et$ of all sets. We denote by $\mathcal{C}^H$ and $\mathcal{C}_H$ the full subcategories of $\mathcal{A}$ consisting of all H-local, respectively H-colocal objects. For objects $X', X \in \mathcal{A}$, we say that X' is a *retract* of X if there are maps $\alpha : X' \to X$ and $\beta : X \to X'$ in $\mathcal{A}$ such that $\beta\alpha = 1_{X'}$. We record without proof the following properties relative to the above considered notions:

Lemma 1.1. *The following hold:*

a) $\operatorname{Eq}(H)$ *is closed under retracts in* $\mathcal{A}^{\to}$.
b) $\operatorname{Eq}(H)$ *satisfies the 'two out of three' property, namely if* $\alpha, \beta \in \mathcal{A}^{\to}$ *are composable morphisms, then if two of the morphisms* $\alpha, \beta, \beta\alpha$ *are H-equivalences, then so is the third.*
c) *The subcategory* $\mathcal{C}^H$ *(respectively* $\mathcal{C}_H$*) is closed under limits (respectively colimits) and both are closed under retracts in* $\mathcal{A}$.

Moreover if every object of $\mathcal{A}$ has a left (right) approximation with an H-local (colocal) object, in a sense becoming precise in the hypothesis of the lemma below, then we are in the situation of a localization (colocalization) functor, as it may be seen from:

Lemma 1.2. *If for every $X \in \mathcal{A}$ there is an H-equivalence $X \to X^H$ with $X^H \in \mathcal{C}^H$ (respectively, $X_H \to X$ with $X_H \in \mathcal{C}_H$), then the assignment $X \mapsto X^H$ ($X \mapsto X_H$) is functorial and defines a left (right) adjoint of the inclusion functor $\mathcal{C}^H \to \mathcal{A}$ ($\mathcal{C}_H \to \mathcal{A}$). Moreover the left (right) adjoint of the inclusion functor sends every map $\alpha \in \mathrm{Eq}(H)$ to an isomorphism and it is universal relative to this property, so $\mathcal{C}^H$ ($\mathcal{C}_H$) is equivalent to the category of fractions of $\mathcal{A}$ with respect to $\mathrm{Eq}(H)$.*

Proof. Straightforward. (The first statement was also noticed in [6, 1.6]). $\square$

In the sequel we consider a pair of adjoint functors $H : \mathcal{A} \to \mathcal{B}$ at the right and $T : \mathcal{B} \to \mathcal{A}$ at the left, where $\mathcal{A}$ and $\mathcal{B}$ are arbitrary categories. We shall symbolize this situation by $T \dashv H$. Consider also the arrows of adjunction

$$\delta : T \circ H \to \mathbf{1}_{\mathcal{A}} \text{ and } \eta : \mathbf{1}_{\mathcal{B}} \to H \circ T.$$

Note that, for all $X \in \mathcal{A}$ and all $Y \in \mathcal{B}$ we obtain the commutative diagrams in $\mathcal{B}$ and $\mathcal{A}$ respectively:

$$(1) \qquad \begin{tikzcd} H(X) \arrow[r, "\eta_{H(X)}"] \arrow[dr, equal, "1_{H(X)}"'] & (H \circ T \circ H)(X) \arrow[d, "H(\delta_X)"] \\ & H(X) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T(Y) \arrow[r, "T(\eta_Y)"] \arrow[dr, equal, "1_{T(Y)}"'] & (T \circ H \circ T)(Y) \arrow[d, "\delta_{T(Y)}"] \\ & T(Y) \end{tikzcd}$$

showing that $H(X)$ and $T(Y)$ are retracts of $(H \circ T \circ H)(X)$, respectively $(T \circ H \circ T)(Y)$. Corresponding to the adjoint pair considered above, we define the following full subcategories of $\mathcal{A}$ and $\mathcal{B}$:

$$\mathcal{S}_H = \{X \in \mathcal{A} \mid \delta_X : (T \circ H)(X) \to X \text{ is an isomorphism}\},$$

and respectively

$$\mathcal{S}^T = \{Y \in \mathcal{B} \mid \eta_Y : Y \to (H \circ T)(Y) \text{ is an isomorphism}\}.$$

The objects in $\mathcal{S}_H$ and $\mathcal{S}^T$ are called δ-*reflexive*, respectively η-*reflexive*. Note that H and T restrict to mutually inverse equivalences of categories between $\mathcal{S}_H$ and $\mathcal{S}^T$ and these subcategories are the largest of $\mathcal{A}$ and $\mathcal{B}$ respectively, enjoying this property. Note that these subcategories were intensively studied in the following particular case: Let R be a ring with identity and let A be a right R–module. If $E = \mathrm{End}_R(A)$ is the endomorphism ring of A, then A has a natural structure of an $E-R$–bimodule, and it induces a pair of adjoint functors $H_A : \text{Mod-}R \leftrightarrows \text{Mod-}E : T_A$ given by $H_A(X) = \mathrm{Hom}_R(A, X)$ and $T_A(Y) = Y \otimes_E A$. Modules in $\mathcal{S}_T$ and $\mathcal{S}^T$ are called reflexive in [3, Section 2.1] (from where our terminology) or static respectively adstatic in [1]. Note also that this last cited paper works in a setting more general that those of module categories, namely in the setting of Grothendieck abelian categories.

Lemma 1.3. *The following inclusions hold:*

a) $\mathcal{S}_H \subseteq \operatorname{Im} T \subseteq \mathcal{C}_H \subseteq \mathcal{A}$.
b) $\mathcal{S}^T \subseteq \operatorname{Im} H \subseteq \mathcal{C}^T \subseteq \mathcal{B}$.

Proof. a) The first inclusion is obvious. For the second inclusion observe that for all $\epsilon \in \mathrm{Eq}(H)$ and all $Y \in \mathcal{B}$ the isomorphism in $\mathcal{S}et^{\to}$

$$\epsilon^* = \mathcal{A}(T(Y), \epsilon) \cong \mathcal{B}(Y, H(\epsilon))$$

shows that ϵ^* is bijective. The inclusions from b) follow by duality. $\square$

Lemma 1.4. *Let $\mathcal{C}$ be a subcategory of $\mathcal{A}$ such that the inclusion functor $I : \mathcal{C} \to \mathcal{A}$ has a right adjoint $R : \mathcal{A} \to \mathcal{C}$ and the arrow of the adjunction $\mu_X : (I \circ R)(X) \to X$ is an H-equivalence for all $X \in \mathcal{A}$. Then μ_X is an isomorphism for all $X \in \mathcal{C}_H$, and consequently $\mathcal{C}_H \subseteq \mathcal{C}$.*

Proof. Let $X \in \mathcal{C}_H$. Since $\mu_X \in \mathrm{Eq}(H)$, we deduce that the induced map

$$\mu_X^* : \mathcal{A}(X, (I \circ R)(X)) \to \mathcal{A}(X, X)$$

is bijective, consequently there is a morphism $\mu'_X : X \to (I \circ R)(X)$ such that $\mu_X \mu'_X = 1_X$. Since $R \circ I \cong \mathbf{1}_{\mathcal{C}}$ naturally, and μ is also natural, we obtain a commutative diagram

$$\begin{tikzcd}
(I \circ R)(X) \arrow[r, "(I\circ R)(\mu'_X)"] \arrow[d, "\mu_X"'] \arrow[dr, equal] & (I \circ R \circ I \circ R)(X) \arrow[d, "\mu_{(I\circ R)(X)}"] \\
X \arrow[r, "\mu'_X"'] & (I \circ R)(X)
\end{tikzcd}$$

showing that $\mu'_X \mu_X = 1_{(I\circ R)(X)}$, hence μ_X is an isomorphism. $\square$

2. Some equivalences induced by adjoint functors

In this section we fix a pair of adjoint functors $T \dashv H$ between two arbitrary categories $\mathcal{A}$ and $\mathcal{B}$, as in Section 1.

Recall that a *monad* on the category $\mathcal{B}$ is a triple (E, η, μ), consisting of an endofunctor $E : \mathcal{B} \to \mathcal{B}$ and two natural transformations $\eta : \mathbf{1}_{\mathcal{B}} \to E$ and $\mu : E \circ E \to E$ inducing the commutative diagrams:

$$\begin{tikzcd}
E \circ E \circ E \arrow[r, "E\mu"] \arrow[d, "\mu E"'] & E \circ E \arrow[d, "\mu"] \\
E \circ E \arrow[r, "\mu"'] & E
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
E \arrow[r, "E\eta"] \arrow[dr, equal] & E \circ E \arrow[d, "\mu"] & E \arrow[l, "\eta E"'] \arrow[dl, equal] \\
& E &
\end{tikzcd}.$$

The monad above is called idempotent if $\mu : E \circ E \to E$ is an isomorphism (see [5, Section 2]). Note that the pair of adjoint functors $T \dashv H$ above induces a monad $(H \circ T, \eta, H\delta T)$ on $\mathcal{B}$. Conversely from a monad we obtain a pair of adjoint functors (see [10, Chapter VI] for details concerning monads).

Theorem 2.1. *The following are equivalent:*

(i) $\mathcal{S}_H = \operatorname{Im} T$.

(ii) $\mathcal{S}^T = \mathcal{C}^T$.
(iii) $\mathcal{S}^T = \operatorname{Im} H$.
(iv) $\mathcal{S}_H = \mathcal{C}_H$.
(v) *The functors H and T induce mutually inverse equivalence of categories between $\mathcal{C}_H$ and $\mathcal{C}^T$.*

Proof. (i)$\Rightarrow$(ii). The condition (i) is equivalent to the fact that the monad $(H \circ T, \eta, H\delta T)$ is idempotent by [5, Proposition 2.1]. Thus $\mathcal{S}^T = \mathcal{C}^T$, by [5, Lemma 2.8] (there the T-local objects are called Eq(T)-closed).

The implication (ii)$\Rightarrow$(iii) follows immediately from Lemma 1.3, whereas the implications (iii)$\Rightarrow$(iv) and (iv)$\Rightarrow$(i) are the dual of (i)$\Rightarrow$(ii), respectively (ii)$\Rightarrow$(iii).

Finally the equivalent conditions (i)–(iv) are also equivalent to (v), because $\mathcal{S}_H$ and $\mathcal{S}^T$ are the largest subcategories of $\mathcal{A}$ and $\mathcal{B}$ for which H and T restrict to mutually inverse equivalences. □

Corollary 2.2. *The adjoint functors $T \dashv H$ induce mutually inverse equivalences $\mathcal{C}_H \rightleftarrows \mathcal{B}$ if and only if T is fully faithful. Dually the adjoint pair induces equivalences $\mathcal{A} \rightleftarrows \mathcal{C}^T$ if and only if H is fully faithful.*

Proof. The functor T is fully faithful exactly if the unit of the adjunction $\eta : \mathbf{1}_{\mathcal{B}} \to (H \circ T)$ is an isomorphism, or equivalently, $\mathcal{S}^T = \mathcal{B}$. Now, Theorem 2.1 applies. □

Theorem 2.1 and Corollary 2.2 generalize [1, Theorem 1.6 and Corollary 1.7], where the work is done in the setting of abelian categories, and the proof stresses the abelian structure. These results may be also compared with [12, Theorem 1.18], where the framework is also that of abelian categories.

Remark 2.3. Most of the implications from Theorem 2.1 are known to specialists. Moreover there are also other categories equivalent to $\mathcal{C}_H$ or $\mathcal{C}^T$. For example they are equivalent to some categories of fractions as in Lemma 1.2 (see also [5, Theorem 2.6]); precisely Lemma 1.2 tells us that in the hypotheses of Theorem 2.1 the functor $\mathcal{A} \to \mathcal{C}_H$, $X \mapsto (T \circ H)(X)$ is the right adjoint of the inclusion functor of $\mathcal{S}_H = \mathcal{C}_H$ into $\mathcal{A}$ (we are in the situation of a colocalization) and the functor $\mathcal{B} \to \mathcal{C}^T$, $Y \mapsto (H \circ T)(Y)$ is the left adjoint of the inclusion functor of $\mathcal{S}^T = \mathcal{C}^T$ into $\mathcal{B}$ (a localization). Other categories which are equivalent to $\mathcal{C}_H$ or $\mathcal{C}^T$ are the so called Eilenberg–Moore category or to the Kleisli category (see [5, Theorems 2.6 and 2.7], results revisited also in [2]). However the equivalence of all five conditions of the theorem above seems to be stated here for the first time.

We consider also the following subcategories of $\mathcal{A}$ and $\mathcal{B}$ respectively:

$$\mathcal{G}_H = \{X \in \mathcal{A} \mid \delta_X : (T \circ H)(X) \to X \text{ is an epimorphism}\},$$

$$\mathcal{G}^T = \{Y \in \mathcal{B} \mid \eta_Y : Y \to (H \circ T)(Y) \text{ is a monomorphism}\}.$$

The dual character of all considerations in the present section continues to hold for $\mathcal{G}_H$ and $\mathcal{G}^T$.

Lemma 2.4. *The following statements hold:*

a) *The subcategory $\mathcal{G}_H$ (respectively $\mathcal{G}^T$) is closed under quotient objects (subobjects).*
b) $\operatorname{Im} T \subseteq \mathcal{G}_H$ *(respectively* $\operatorname{Im} H \subseteq \mathcal{G}^T$*).*

Proof. a) Let $\alpha : X' \to X$ be an epimorphism in $\mathcal{A}$ with $X' \in \mathcal{G}_H$. Since δ is natural, we obtain the equality $\alpha\delta_{X'} = \delta_X(T \circ H)(\alpha)$, showing that δ_X is an epimorphism together with $\alpha\delta_{X'}$.

b) From the diagrams (1), we see that $\delta_{T(Y)}$ is right invertible, so it is an epimorphism for any $Y \in \mathcal{B}$. Thus $\operatorname{Im} T \subseteq \mathcal{G}_H$. □

The subcategory $\mathcal{G}_H$ of $\mathcal{A}$ is more interesting in the case when $\mathcal{A}$ has *epimorphic images*, what means that it has images and the factorization of a morphism through its image is a composition of an epimorphism followed by a monomorphism (for example, $\mathcal{A}$ has epimorphic images, provided that it has equalizers and images, by [11, Chapter 1, Proposition 10.1]). Suppose also that $\mathcal{A}$ is *balanced*, that is every morphism which is both epimorphism and monomorphism is an isomorphism. Thus every factorization of a morphism as a composition of an epimorphism followed by a monomorphism is a factorization through image, by [11, Chapter 1, Proposition 10.2]. With these hypotheses it is not hard to see that the factorization of a morphism through its image is functorial, that means the assignment $\alpha \mapsto \operatorname{im}\alpha$ defines a functor $\mathcal{A}^{\to} \to \mathcal{A}$. Note that a category $\mathcal{A}$ is balanced with epimorphic images if and only if its dual satisfies the same property, so we may dualize results established in such categories.

Proposition 2.5. *If $\mathcal{A}$ is a balanced category with epimorphic images, then the functor $\mathcal{A} \to \mathcal{G}_H$, $X \mapsto \operatorname{im}\delta_X$ is a right adjoint of the inclusion functor $\mathcal{G}_H \to \mathcal{A}$.*

Proof. By hypothesis $\operatorname{im}\delta_X$ is a quotient of $(H \circ T)(Y)$ and $H(T(Y)) \in \mathcal{G}_H$, so the functor $\mathcal{A} \to \mathcal{G}_H$, $X \mapsto \operatorname{im}\delta_X$ is well defined, by Lemma 2.4. Let now $\alpha : X' \to X$ be in $\mathcal{A}^{\to}$, where $X' \in \mathcal{G}_H$ and $X \in \mathcal{A}$. Since $\delta_{X'}$ is an epimorphism, it follows that

$$\operatorname{im}\alpha = \operatorname{im}(\alpha\delta_{X'}) = \operatorname{im}(\delta_X(T \circ H)(\alpha)) \subseteq \operatorname{im}\delta_X,$$

so α factors through $\operatorname{im}\delta_X$. This means that the map

$$\mathcal{A}(X', \operatorname{im}\delta_X) \to \mathcal{A}(X', X)$$

is surjective. But it is also injective since the functor $\mathcal{A}(X', -)$ preserves monomorphisms, and the conclusion follows. □

Corollary 2.6. *If $\mathcal{A}$ is a balanced category with epimorphic images, then the morphism $\operatorname{im}\delta_X \to X$ is an H-equivalence and $\mathcal{C}_H \subseteq \mathcal{G}_H$.*

Proof. The second statement of the conclusion follows from the first one by using Proposition 2.5 and Lemma 1.4. But H carries the monomorphism

$\operatorname{im}\delta_X \to X$ to a monomorphism in $\mathcal{B}$, because H is a right adjoint. Moreover, since $H(\delta_X)$ is right invertible, the same is true for the morphism $H(\operatorname{im}\delta_X) \to H(X)$, as we may see from the commutative diagram

$$\begin{tikzcd}
(H \circ T \circ H)(X) \arrow[rr, "H(\delta_X)"] \arrow[dr] & & H(X) \\
& H(\operatorname{im}\delta_X) \arrow[ur] &
\end{tikzcd}.$$

□

Proposition 2.7. *Suppose $\mathcal{B}$ is a balanced category with epimorphic images. The following are equivalent:*

(i) *The pair of adjoint functors $T \dashv H$ induces mutually inverse equivalences $\mathcal{C}_H \rightleftarrows \mathcal{G}^T$.*
(ii) *$\eta_Y : Y \to (H \circ T)(Y)$ is an epimorphism for all $Y \in \mathcal{B}$.*

Proof. (i)⇒(ii). Denote $Y' = \operatorname{im}\eta_Y$. Then the unit η_Y of adjunction factors as $Y \to Y' \to (H \circ T)(Y)$, where the epimorphism $Y \to Y'$ is a T-equivalence by the dual of Corollary 2.6, and $Y' \to (H \circ T)(Y)$ a monomorphism. Since $(H \circ T)(Y) \in \operatorname{Im} H \subseteq \mathcal{G}^T$ and $\mathcal{G}^T$ is closed under subobjects, we deduce that $Y' \in \mathcal{G}^T$. Now (i) implies that $\eta_{Y'}$ is an isomorphism, so the diagram

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, "\eta_Y"'] & Y' \arrow[d, "\eta_{Y'}"] \\
(H \circ T)(Y) \arrow[r, "\cong"] & (H \circ T)(Y')
\end{tikzcd}$$

proves (ii).

(ii)⇒(i). The condition (ii) implies that the monad $(H \circ T, \eta, H\delta T)$ is idempotent, according to [2, 3.7]. Thus $\operatorname{Im} T = \mathcal{S}_H$ and (i) follows by Theorem 2.1. □

Combining Proposition 2.7 and its dual we obtain:

Theorem 2.8. *Suppose both $\mathcal{A}$ and $\mathcal{B}$ are balanced categories with epimorphic images. The following are equivalent:*

(i) *The pair of adjoint functors $T \dashv H$ induces mutually inverse equivalences $\mathcal{G}_H \rightleftarrows \mathcal{G}^T$.*
(ii) *$\delta_X : (T \circ H)(X) \to X$ is a monomorphism for all $X \in \mathcal{A}$ and $\eta_Y : Y \to (H \circ T)(Y)$ is an epimorphism for all $Y \in \mathcal{B}$.*

Remark that [4, Proposition 2.2.4 and Theorem 2.3.8] provide characterizations of (weak) $*$-modules which are analogous to Proposition 2.7 and Theorem 2.8 above. These results will be used in Section 3, for defining the corresponding notions in a non additive situation. We note also that the implications (ii)⇒(i) in both Proposition 2.7 and Theorem 2.8 are already considered in a more general situation in the recent preprint [2].

We end this section with a better determination of the category $\mathcal{G}_H$ given in a particular case, a result which we need in the following section. Suppose now $\mathcal{A}$ is a cocomplete category and $\mathcal{E}$ is a small category. Denote by $[\mathcal{E}^{\rm op}, \mathcal{S}et]$ the category of all contravariant functors from $\mathcal{E}$ into $\mathcal{S}et$. Then we view $\mathcal{E}$ as a subcategory of $[\mathcal{E}^{\rm op}, \mathcal{S}et]$, via the Yoneda embedding $\mathcal{E} \to [\mathcal{E}^{\rm op}, \mathcal{S}et]$, $e \mapsto \mathcal{E}(-, e)$. For simplicity, we shall write $[Y', Y]$ for $[\mathcal{E}^{\rm op}, \mathcal{S}et](Y', Y)$, where $Y', Y \in [\mathcal{E}^{\rm op}, \mathcal{S}et]$. For every $Y \in [\mathcal{E}^{\rm op}, \mathcal{S}et]$ denote by $\mathcal{E} \downarrow Y$ the comma category whose objects are of the form (e, y) with $e \in \mathcal{E}$ and $y \in Y(e)$ and whose morphisms are

$$(\mathcal{E} \downarrow Y)((e', y'), (e, y)) = \{\alpha \in \mathcal{E}(e', e) \mid Y(\alpha)(y') = y\}.$$

The projection functor $\mathcal{E} \downarrow Y \to \mathcal{E}$ is given by $(e, y) \mapsto e$ and $\alpha \mapsto \alpha$ for all $(e, y) \in \mathcal{E} \downarrow Y$ and all $\alpha \in (\mathcal{E} \downarrow Y)((e', y'), (e, y))$. Observe then that the subcategory $\mathcal{E}$ is *dense* in $[\mathcal{E}^{\rm op}, \mathcal{S}et]$, what means, for every $Y \in [\mathcal{E}^{\rm op}, \mathcal{S}et]$ it holds

$$Y \cong \operatorname{colim}((\mathcal{E} \downarrow Y) \to \mathcal{E} \to [\mathcal{E}^{\rm op}, \mathcal{S}et]) = \operatorname*{colim}_{(e,y)\in\mathcal{E}\downarrow Y} \mathcal{E}(-, e),$$

where the last notation is a shorthand for the previous colimit.

For a functor $A : \mathcal{E} \to \mathcal{A}$, consider the left Kan extension of A along the Yoneda embedding:

$$T_A : [\mathcal{E}^{\rm op}, \mathcal{S}et] \to \mathcal{A}, T_A(Y) = \operatorname*{colim}_{(e,y)\in\mathcal{E}\downarrow Y} A(e),$$

which may be characterized as the unique, up to a natural isomorphism, colimit preserving functor $[\mathcal{E}^{\rm op}, \mathcal{S}et] \to \mathcal{A}$, mapping $\mathcal{E}(-, e)$ into $A(e)$ for all $e \in \mathcal{E}$. The functor T_A has a right adjoint, namely the functor

$$H_A : \mathcal{A} \to [\mathcal{E}^{\rm op}, \mathcal{S}et], H_A(X) = \mathcal{A}(A(-), X).$$

In order to use the results of Section 1, we recall the notations made there, namely let $\delta : T_A \circ H_A \to \mathbf{1}_\mathcal{A}$ and $\eta : \mathbf{1}_\mathcal{B} \to H_A \circ T_A$ be the arrows of the adjunction. For simplicity we shall replace in the next considerations the subscript H_A and the superscript T_A with A. So objects in $\mathcal{C}_A$, $\mathcal{C}^A$, $\mathcal{G}_A$ and $\mathcal{G}^A$ will be called *A-colocal*, *A-local*, *A-generated*, respectively *A-cogenerated*.

Suppose additionally that A is fully faithful. Note that this additional assumption means that the category $\mathcal{E}$ may be identified with a (small) subcategory of $\mathcal{A}$ and A with the inclusion functor. For example, if $\mathcal{E}$ has a single object, then A may be identified with an object of $\mathcal{A}$.

Lemma 2.9. *If $\mathcal{A}$ is a cocomplete, balanced category with epimorphic images and $A : \mathcal{E} \to \mathcal{A}$ is fully faithful, then it holds:*

a) *$A(e) \in \mathcal{S}_A$ for all $e \in \mathcal{E}$.*
b) *An object $X \in \mathcal{A}$ is A-generated exactly if there is an epimorphism $A' \to X$ with A' a coproduct of objects of the form $A(e)$ with $e \in \mathcal{E}$.*

Proof. a) Since A is fully faithful, we have the natural isomorphisms:

$$(T_A \circ H_A)(A(e)) = T_A(\mathcal{A}(A(-), A(e)) \cong T_A(\mathcal{E}(-, e)) \cong A(e),$$

for every $e \in \mathcal{E}$.

b) Let $A' = \coprod A(e_i) \in \mathcal{A}$ be a coproduct of objects of the form $A(e)$. By the result in a) we deduce

$$A' = \coprod A(e_i) \cong \coprod (T_A \circ H_A)(A(e_i)) \cong T_A(\coprod H_A(A(e_i))) \in \operatorname{Im} T_A,$$

so $A' \in \mathcal{G}_A$, since $\operatorname{Im} T_A \subseteq \mathcal{G}_A$, an inclusion established in Lemma 2.4. If $X \in \mathcal{A}$ such that there is an epimorphism $A' \to X$, then $X \in \mathcal{G}_A$, again by Lemma 2.4. Conversely, for every $X \in \mathcal{A}$, the object $H_A(X)$ of $[\mathcal{E}^{\mathrm{op}}, \mathcal{S}et]$ may be written as

$$H_A(X) \cong \operatorname*{colim}_{(e,x)\in \mathcal{E}\downarrow H_A(X)} \mathcal{E}(-,e) = \operatorname*{colim}_{(e,x)\in A\downarrow X} \mathcal{E}(-,e),$$

where the comma category $A \downarrow X$ has as objects pairs of the form (e,x) with $e \in \mathcal{E}$ and $x \in \mathcal{A}(A(e), X)$. Thus

$$(T_A \circ H_A)(X) \cong \operatorname*{colim}_{(e,x)\in A\downarrow X} T_A(\mathcal{E}(-,e)) \cong \operatorname*{colim}_{(e,x)\in A\downarrow X} A(e),$$

so there is an epimorphism from $\coprod_{(e,x)\in A\downarrow X} A(e)$ to $(T_A \circ H_A)(X)$. Further the morphism $\delta_X : (T_A \circ H_A)(X) \to X$ is an epimorphism too, for $X \in \mathcal{G}_A$. Composing them we obtain the desired epimorphism. □

3. $*$-acts over monoids

We see a monoid M as a category with one object whose endomorphism set is M. We consider the category $[M^{\mathrm{op}}, \mathcal{S}et]$ of all contravariant functors from this category to the category of sets, and we call it the category of *(right) acts* over M, or simply *M-acts*. Clearly an M-act is a set X together with an action $X \times M \to X$, $(x,m) \mapsto xm$ such that $(xm)m' = x(mm')$ and $x1 = x$ for all $x \in X$ and all $m, m' \in M$. Left acts are covariant functors $M \to \mathcal{S}et$, that is sets X together with an action $M \times X \to X$, satisfying the corresponding axioms. For the general theory of acts over monoids and undefined notions concerning this subject we refer to [9]. We should mention here that in contrast with [9] we allow the empty act to be an object in our category of acts, for the sake of (co)completness. Note that the category of M-acts is balanced and has epimorphic images, by [9, Proposition 1.6.15 and Theorem 1.4.21].

Fix a monoid M and an object $A \in [M^{\mathrm{op}}, \mathcal{S}et]$. In order to use the results of the preceding sections, we identify A with a fully faithful functor $E \to [M^{\mathrm{op}}, \mathcal{S}et]$ where E is the endomorphism monoid of A. Thus A is canonically an $E - M$–biact (see [9, Definition 1.4.24]), so we obtain two functors

$$H_A : [M^{\mathrm{op}}, \mathcal{S}et] \to [E^{\mathrm{op}}, \mathcal{S}et],\ H_A(X) = [A, X]$$

and

$$T_A : [E^{\mathrm{op}}, \mathcal{S}et] \to [M^{\mathrm{op}}, \mathcal{S}et],\ T_A(Y) = Y \otimes_E A$$

the second one being the left adjoint of the first (see [9, Definition 2.5.1 and Proposition 2.5.19]). Clearly these functors agree with the functors defined at the end of Section 2.

We say that A is a *(weak) $*$-act* if the above adjoint pair induces mutually inverse equivalences $H_A : \mathcal{G}_A \rightleftarrows \mathcal{G}^A : T_A$ (respective $H_A : \mathcal{C}_A \rightleftarrows \mathcal{G}^A : T_A$). Note that our definitions for subcategories $\mathcal{G}_A$ and $\mathcal{G}^A$ agree with the characterizations of all A-generated respectively A^*-cogenerated modules given in [3, Lemma 2.1.2]. As we may see from Proposition 2.7, our subcategory $\mathcal{C}_A$ seems to be the non–additive counterpart of the subcategory of all A-presented modules (compare with [3, Proposition 2.2.4]).

In what follows, we need more definitions relative to an M-act A. First A is called *decomposable* if there exist two non empty subacts $B, C \subseteq A$ such that $A = B \cup C$ and $B \cap C = \emptyset$ (see [9, Definition 1.5.7]). In this case $A = B \sqcup C$, since coproducts in the category of acts are the disjoint unions, by [9, Proposition 2.1.8]. If A is not decomposable, then it is called *indecomposable*. Second, A is said to be *weak self–projective* provided that $(H_A \circ T_A)(g)$ is an epimorphism whenever $g : U \to Y$ is an epimorphism in $[E^{\mathrm{op}}, \mathcal{S}et]$ with $U \in \mathcal{S}^A$. More explicitly, if $g : U \to Y$ is an epimorphism in $[E^{\mathrm{op}}, \mathcal{S}et]$, then $T_A(g)$ is an epimorphism in $[M^{\mathrm{op}}, \mathcal{S}et]$ and our definition requires that A is projective relative to such epimorphisms for which $U \in \mathcal{S}^A$. Third A is called *(self–)small* provided that the functor H_A preserves coproducts (of copies of A).

Lemma 3.1. *With the notations above, the following are equivalent:*

(i) *A is small.*
(ii) *A is self–small.*
(iii) *$E^{(I)}$ is η-reflexive for any set I, where $E^{(I)}$ denotes the coproduct indexed over I of copies of E.*
(iv) *$E \sqcup E$ is η-reflexive.*
(v) *A is indecomposable.*

Proof. (i)⇒(ii) is obvious.

(ii)⇒(iii). If H_A commutes with coproducts of copies of A then

$$\begin{aligned} E^{(I)} = \coprod_I [A, A] &\cong \left[A, \coprod_I A\right] \cong \left[A, \coprod_I (E \otimes_E A)\right] \\ &\cong \left[A, \left(E^{(I)}\right) \otimes_E A\right] \cong (H_A \circ T_A)\left(E^{(I)}\right). \end{aligned}$$

(iii)⇒(iv) is obvious.

(iv)⇒(v). If A is decomposable, that is $A = B \sqcup C$ with $B \neq \emptyset$ and $C \neq \emptyset$, then let $i_B : B \to A$ and $i_C : C \to A$ be the canonical injections of this coproduct. Denote also by $j_1, j_2 : A \to A \sqcup A$ the corresponding canonical injections. The homomorphisms of M-acts $j_1 i_B : B \to A \sqcup A$ and $j_2 i_C : C \to A \sqcup A$ induce a unique homomorphism $f : A = B \sqcup C \to A \sqcup A$. Obviously $f \in (H_A \circ T_A)(E \sqcup E)$ but $f \notin [A, A] \sqcup [A, A] = E \sqcup E$.

(v)⇒(i) is [9, Lemma 1.5.37].

□

Proposition 3.2. *The following statements hold:*

a) *If A is a weak $*$-act then A is weak self–projective.*
b) *If A is weak self–projective and indecomposable, then A is a weak $*$-act.*

Proof. a) Let A be a weak $*$-act and let $g : U \to Y$ be an epimorphism in $[E^{\rm op}, \mathcal{S}et]$ with $U \in \mathcal{S}^A$. We know by Proposition 2.7 that η_Y is epic, and by the naturalness of η that $(H_A \circ T_A)(g)\eta_U = \eta_Y g$. Since $\eta_Y g$ is an epimorphism we deduce that $(H_A \circ T_A)(g)$ is an epimorphism too.

b) As we have already noticed H_A preserves coproducts, provided that A is indecomposable. Thus $\mathcal{S}^A$ is closed under arbitrary coproducts in the category of E-acts. For a fixed $Y \in [E^{\rm op}, \mathcal{S}et]$ there is an epimorphism $g : E^{(I)} \to Y$. Since E is η-reflexive the same is also true for $E^{(I)}$. But $(H_A \circ T_A)(g)$ is an epimorphism, since A is weak self–projective. From the equality $(H_A \circ T_A)(g)\eta_{E^{(I)}} = \eta_Y g$ it follows that η_Y is an epimorphism too. The conclusion follows by Proposition 2.7. □

Theorem 3.3. *The following statements hold:*

a) *If A is a $*$-act then A is weak self–projective and $\mathcal{C}_A = \mathcal{G}_A$.*
b) *If A is indecomposable, weak self–projective and $\mathcal{C}_A = \mathcal{G}_A$, then A is a $*$-act.*

Proof. Both implications follow at once from Proposition 3.2. □

Remark 3.4. Propositions 2.7 and 3.2 and Theorems 2.8 and 3.3 provide a non additive version of [3, Proposition 2.2.4] respectively [3, Theorem 2.3.8]. In contrast with the case of modules, where the functors are additive, for acts it is not clear that a weak $*$-object must be indecomposable (the non additive version of self–smallness as we have seen in Lemma 3.1). The main obstacle for deducing this implication in the new setting comes from the fact that non additive functors do not have to preserves finite coproducts.

Using the characterization of so called tilting modules given in [3, Theorem 2.4.5], we may define a *tilting* M-act to be a $*$-act A such that the injective envelope of M belongs to $\mathcal{G}_A$. Note that injective envelopes exist in $[M^{\rm op}, \mathcal{S}et]$ by [9, Corollary 3.1.23]. We may ask ourselves if some results which are known for tilting modules, e.g. the so called tilting theorem [3, Theorem 3.5.1], do have correspondents for acts. We don't answer now to this question, the study of this parallelism remaining as a subject for future research.

Our next aim is to infer from our results the Morita–type theorem for monoids (see [9, Section 5.3]). In order to perform it we need a couple of lemmas.

Lemma 3.5. *If the M-act A is a generator in $[M^{\rm op}, \mathcal{S}et]$ then $\mathcal{C}_A = \mathcal{G}_A = [M^{\rm op}, \mathcal{S}et]$.*

Proof. For a generator A of $[M^{\mathrm{op}}, \mathcal{S}et]$ the equality $\mathcal{G}_A = [M^{\mathrm{op}}, \mathcal{S}et]$ follows by Lemma 2.9. Moreover M is a retract of A by [9, Theorem 2.3.16], therefore $M \in \mathcal{C}_A$, since $\mathcal{C}_A$ is closed under retracts. Thus a morphism $\epsilon : U \to V$ in $[M^{\mathrm{op}}, \mathcal{S}et]$ is an A-equivalence if and only if it is an isomorphism, therefore $\mathcal{C}_A = [M^{\mathrm{op}}, \mathcal{S}et]$. □

Recall that the left E-act A is said to be *pullback flat* if the functor $T_A = (- \otimes_E A)$ commutes with pullbacks (see [9, Definition 3.9.1]).

Lemma 3.6. *If the right M-act A is indecomposable, weak self projective and the left E-act A is pullback flat, then $\mathcal{G}^A = [E^{\mathrm{op}}, \mathcal{S}et]$.*

Proof. First observe that A is a weak $*$-act by Proposition 3.2. Hence $\mathcal{G}^A = \mathcal{C}^A = \mathcal{S}^A$, and this subcategory has to be closed under subacts and limits. Moreover $E^{(I)}$ is η-reflexive for any set I according to Lemma 3.1. For a fixed $Y \in [E^{\mathrm{op}}, \mathcal{S}et]$ there is an epimorphism $g : E^{(I)} \to Y$. Take the kernel pair of g, that is construct the pullback

$$\begin{tikzcd} K \arrow[r, "k_1"] \arrow[d, "k_2"'] & E^{(I)} \arrow[d, "g"] \\ E^{(I)} \arrow[r, "g"'] & Y \end{tikzcd} .$$

The functors T_A and H_A preserve pullbacks, the first one by hypothesis and the second one automatically. Moreover K is a subact of $E^{(I)} \times E^{(I)}$ and the closure properties of $\mathcal{S}^A$ imply that $K \cong (T_A \circ H_A)(K)$. Applying the functor $H_A \circ T_A$ to the above diagram and having in mind the previous observations we obtain a pullback diagram

$$\begin{tikzcd} K \arrow[r, "k_1"] \arrow[d, "k_2"'] & E^{(I)} = (H_A \circ T_A)\left(E^{(I)}\right) \arrow[d, "(H_A \circ T_A)(g)"] \\ E^{(I)} = (H_A \circ T_A)\left(E^{(I)}\right) \arrow[r, "(H_A \circ T_A)(g)"'] & (H_A \circ T_A)(Y) \end{tikzcd} .$$

Note that $(H_A \circ T_A)(g)$ is an epimorphism by hypothesis. Then we know by [9, Theorem 2.2.44] that both g and $(H_A \circ T_A)(g)$ are coequalizers for the pair (k_1, k_2). Therefore we deduce that $Y \cong (H_A \circ T_A)(Y)$ canonically, so $Y \in \mathcal{S}^A$. Thus $\mathcal{G}^A = \mathcal{S}^A = [E^{\mathrm{op}}, \mathcal{S}et]$. □

Now we are ready to prove the desired Morita–type result:

Theorem 3.7. *Let M and E be two monoids. Then the categories $[M^{\mathrm{op}}, \mathcal{S}et]$ and $[E^{\mathrm{op}}, \mathcal{S}et]$ are equivalent via the mutually inverse equivalence functors H and T if and only if there is a cyclic, projective generator A of $[M^{\mathrm{op}}, \mathcal{S}et]$ such that E is the endomorphism monoid of A, in which case $H = H_A$ and $T = T_A$.*

Proof. First note that a projective act is indecomposable if and only if it is cyclic in virtue of [9, Propositions 1.5.8 and 3.17.7].

If $H : [M^{\mathrm{op}}, \mathcal{S}et] \rightleftarrows [E^{\mathrm{op}}, \mathcal{S}et] : T$ are mutually inverse equivalences, then set $A = T(E)$. Thus $T \cong T_A$, as the unique functor, up to a natural isomorphism, which commutes with colimits and satisfies $T_A(E) = A$ (see also [7, Theorem 2.10]), and therefore $H \cong H_A$. Moreover the endomorphism monoid of A is E, and A has to be projective, indecomposable and generator together with E.

Conversely if A is indecomposable and projective in $[M^{\mathrm{op}}, \mathcal{S}et]$ then it is a weak $*$-act by Proposition 3.2. Since A is in addition a generator, Lemma 3.5 tells us that A is a $*$-act and $\mathcal{C}_A = \mathcal{G}_A = [M^{\mathrm{op}}, \mathcal{S}et]$ and Theorem 3.3 implies that A is a $*$-act. Finally the left E-act A is projective by [9, Corollary 3.18.17], so it is strongly flat by [9, Proposition 3.15.5], which means that T_A commutes both with pullbacks and equalizers. Thus $\mathcal{G}^A = [E^{\mathrm{op}}, \mathcal{S}et]$, according to Lemma 3.6. □

References

[1] F. Castaño Iglesias, J. Gómez–Torrecillas, R. Wisbauer, Adjoint functors and equivalences of subcategories, *Bull. sci. math.* 127 (2003), 379–395.

[2] J. Clark, R. Wisbauer, Idempotent monads and $\star$-functors, arXiv:0909.3162v1, available at `http://de.arxiv.org/abs/0909.3162`

[3] R. Colpi, K. Fuller, *Equivalence and Duality for Module Categories*, Cambridge University Press, Cambridge, 2004.

[4] R. Colpi, K. Fuller, Tilting objects in abelian categories and quasitilted rings, *Trans. Amer. Math. Soc.*, 359 (2007), 741–765.

[5] A. Deleanu, A. Frei, P. Hilton, Idempotent triples and completion, Math. Z. 143 (1975), 91-104.

[6] W. Dwyer, Localizations, in *Axiomatic, Enriched and Motivic Homotopy Theory*, Proceedings of the NATO ASI (ed. J.P.C. Greenlees), Kluwer, 2004.

[7] J. Fisher–Palmquist, D. Newell, Triples on functor categories, *J. Algebra*, 25 (1973), 226–258.

[8] G. M. Kelly, *Basic Concepts of Enriched Category Theory*, reprint in *Theory and Applications of Categories*, 10 (2005).

[9] M. Kilp, U. Knauer, A. Mikhalev, *Monoids, Acts and Categories*, De Gruyter Expositions in Mathematics, 29, Walter de Gruyter, Berlin, New–York, 2000.

[10] S. Maclane, *Categories for the Working Mathematician*, Springer Verlag, New York, Heidelberg, Berlin, 1971.

[11] B. Mitchell, *Theory of Categories*, Academic Press, New York and London, 1965.

[12] C. Modoi, Equivalences induced by adjoint functors, *Comm. Alg.* 31(5) (2003), 2327–2355.

[13] N. Popescu, L. Popescu, *Theory of Categories*, Editura Academiei, Bucureşti and Sijthoff & Noordhoff International Publishers, 1979.

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